Probability

1. Suppose we have a coin that when tossed results in heads with probability p, and tails with probability $1-p$. Let X be the number of times that we need to toss the coin until r heads are obtained.

   (a) What is $\mathrm{P}(X = n)$?

   (b) Compute $\mathrm{E}[X]$.

   (c) Compute $\mathrm{Var}(X)$.

   (d) What is $\mathrm{P}(X = n | 1^{\mathrm{st}}$ head occurs on the 5^{th}toss)?

2. Suppose the continuous random variable X has the probability density function

$$f(x) = \begin{cases} c(4x^2 + 6x), & x \in [0, 2] \\ 0, & \text{otherwise} \end{cases}$$

   for some constant c.

   (a) What is $\mathrm{E}[X]$?

   (b) Calculate $\mathrm{P}(X > 1.5)$.

3. Suppose the continuous random vector (X, Y) has the joint probability distribution

$$f(x, y) = \begin{cases} c(4x^2 y + y^2), & x \in [0, 1], y \in [0, 1] \\ 0, & \text{otherwise} \end{cases}$$

   for some constant c.

   (a) Calculate $\mathrm{P}(X + Y > 1.5)$.

   (b) Calculate $\mathrm{E}[Y]$.

   (c) What is $\mathrm{Cov}(X, Y)$?

   (d) What is $f(x|y)$, the conditional probability density function of x given y?

   (e) Compute $\mathrm{E}[X|Y = .5]$.

4. The conditional expectation identity states that for random variables, X and Y, we have

$$\mathrm{E}[X] = \mathrm{E}\left[\mathrm{E}[X|Y]\right].$$

   Similarly, the conditional variance identity states that

$$\mathrm{Var}(X) = \mathrm{Var}(\mathrm{E}[X|Y]) + \mathrm{E}[\mathrm{Var}(X|Y)].$$

   Now let $W = X_1 + X_2 + \ldots + X_n$ where the X_i's are IID and n is also a random variable, independent of the X_i's.

   (a) Use the conditional expectation identity to show that $\mathrm{E}[W] = \mathrm{E}[X_1]\mathrm{E}[n]$.

   (b) Use the conditional variance identity to show that $\mathrm{Var}(W) = \mathrm{E}[X_1^2]\mathrm{Var}(n) + \mathrm{Var}(X_1)\mathrm{E}[n]$.