Probability

1. Suppose we have a coin that when tossed results in heads with probability p, and tails with probability $1-p$. Let X be the number of times that we need to toss the coin until r heads are obtained.

   (a) What is $P(X=n)$?

   (b) Compute $E[X]$.

   (c) Compute $Var(X)$.

   (d) What is $P(X=n|1^{st}$ head occurs on the 5^{th}toss)?

2. Suppose the continuous random variable X has the probability density function

$$f(x) = \begin{cases} c(4x^2+6x), & x \in [0,2] \\ 0, & \text{otherwise} \end{cases}$$

   for some constant c.

   (a) What is $E[X]$?

   (b) Calculate $P(X > 1.5)$.

3. Suppose the continuous random vector (X, Y) has the joint probability distribution

$$f(x,y) = \begin{cases} c(4x^2y+y^2), & x \in [0,1], y \in [0,1] \\ 0, & \text{otherwise} \end{cases}$$

   for some constant c.

   (a) Calculate $P(X+Y > 1.5)$.

   (b) Calculate $E[Y]$.

   (c) What is $Cov(X,Y)$?

   (d) What is $f(x|y)$, the conditional probability density function of x given y?

   (e) Compute $E[X|Y=.5]$.

4. The conditional expectation identity states that for random variables, X and Y, we have

$$E[X] = E\left[E[X|Y]\right].$$

   Similarly, the conditional variance identity states that

$$Var(X) = Var(E[X|Y]) + E[Var(X|Y)].$$

   Now let $W = X_1 + X_2 + \ldots + X_n$ where the X_i's are IID and n is also a random variable, independent of the X_i's.

   (a) Use the conditional expectation identity to show that $E[W] = E[X_1]E[n]$.

   (b) Use the conditional variance identity to show that $Var(W) = E[X_1^2]Var(n) + Var(X_1)E[n]$.

5. Let $X = a + bZ$ where Z is a standard normal random variable and a and b are constants. Write down the probability density and cumulative distribution functions of X. Derive the mean and variance of e^Z and e^X.

Calculus

6. Find $\int x \sin(x)\, dx$.

7. Solve the following ordinary differential equation

$$\frac{d^2y}{dx^2} + 2\frac{dy}{dx} - 3y = 2x - 17.$$

8. A manufacturer of dog food wishes to package the product in cylindrical metal cans, each of which is to contain a certain volume V_0 of food. Find the ratio of the height of the can to its radius when the amount of metal used to make the can is minimized. Assume that the ends and side of the can are made from metal of the same thickness.

9. Compute the derivative with respect to t of $g(t) := \int_0^t f(t,x)dx$.

Linear Algebra

10. Let $A = \begin{bmatrix} 2 & 3 & 7 \\ 4 & 1 & 3 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}$. Is it possible to compute both the products AB and BA? Compute the ones that are possible.

11. Consider the following system of linear equations in the three variables (x_1, x_2, x_3):

$$\begin{aligned} 3x_1 + 7x_2 &= 25 \\ 4x_2 + 7x_3 &= 11 \\ 11x_1 + 3x_3 &= 7 \end{aligned}$$

Represent these equation as a single matrix-vector equation of the form $Ax = b$ where $x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$. Solve the system of equations (you need not use the matrix version of the system).

12. Compute all solutions of the following system of equations

$$\begin{bmatrix} 2 & 3 & 7 \\ 4 & 1 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

13. Solve the following system of equations

$$\begin{bmatrix} 2 & 4 \\ 3 & 1 \\ 7 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 10 \\ 5 \\ 13 \end{bmatrix}$$

Will this system of equations have a solution for all values of the right hand side vector? If not, produce a vector for which this system has no solution? How is the solution of this problem related to the solution of the previous problem?